OUR HERITAGE
ISSN: 0474-9030Vol-68, Special Issue-38
ONE DAY NATIONAL CONFERENCE ON RECENT ADVANCES IN SCIENCES
Held on: 13th February 2020.
Organized by: Department of PHYSICS, CHEMISTRY, MATHEMATICS, BOTANY & ZOOLOGY Shivaji Arts, Commerce and Science College Kannad. Dist: Aurangabad (MS)

MATHEMATICAL APROACH TO RAINBOW

V P Sangale
Head, Dept. of Mathematics, R. B. Attal College, Georai, Dist. Beed (MS) India
E-mail: vpsangale67@gmail.com

Abstract:

The aim of the present paper is to geometrical explanation of reflection and refraction in the apparent position of a rainbow relative to the sun, and mathematics shows why light is concentrated in the rainbow. When rain falls heavily, the rainbow appears opposite of the sun in sky as an arc of spectrum colors. The color separation in a rainbow is due to the fact that the constant k in the law of refraction is slightly different for different colors of light. To calculate the rainbow angle we have to use not only the Snell's law, law of reflection but also the law of sine, geometry and derivative.

Key Words - Snell's law, reflection, refraction, spectrum colors, antisolar point

Introduction:

There are mathematical principles behind phenomena that take place in nature. Such phenomena are take place at the microscopic and sub microscopic levels also from planetary to galactic scales. But trees, leaves, spider webs, bubbles, waves, clouds, rainbows, these are elements of the stuff we can see easily. In garden we can observe the arrangements of leaves, petals, seeds, and flowers are intimately associated with spiral pattern, the related sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … also with an angle of about 137.5^0 . In the sky brightly coloured circular arcs - Rainbows – beautify the sky after rain showers. The rainbow is the most beautiful visual displays in nature. It can be seen, photographed and described as a phenomenon of mathematical physics, but it can not be located at a specific place, only in particular direction. The rainbow can be described as an image of sunlight displaced by reflection and dispersed by refraction in raindrops, seen by an observer with his back to the sun. The geometry of reflection and refraction explains the apparent position of rainbow relative to sun.

Observation of Rainbows:

All rainbows are not alike. Suppose we see rainbow at sunrise or sunset. The rainbow made at this time appears to be a semicircle centred at the point on the horizon opposite to the sun, with the angle of elevation from the observer to the top of the rainbow being about 42^0 .

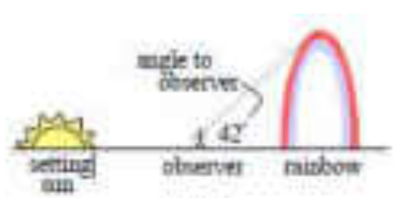


Fig. 1

Earlier in the day, the picture is a little different. The rainbow is at the base of an imaginary cone, whose *axis* is the line from the observer to the sun, and *angle* between side and axis is 42^0.

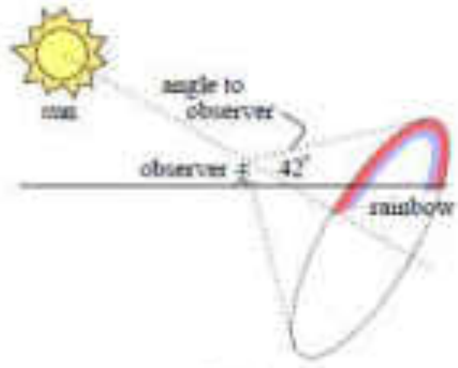
OUR HERITAGE

ISSN: 0474-9030Vol-68, Special Issue-38

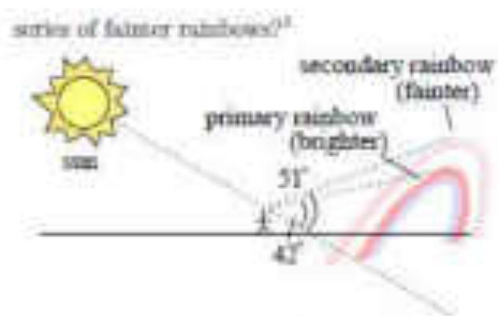
ONE DAY NATIONAL CONFERENCE ON RECENT ADVANCES IN SCIENCES

Held on: 13th February 2020.

Organized by: Department of PHYSICS, CHEMISTRY, MATHEMATICS, BOTANY & ZOOLOGY Shivaji Arts, Commerce and Science College Kannad. Dist: Aurangabad (MS)



The phenomenon which we may have observed is the secondary rainbow. When the conditions for rainbows are especially good, there will be another, fainter rainbow above the primary rainbow. The angle of elevation of the secondary rainbow is about 51^0. If the conditions are extremely good, might it be possible to see still fainter tertiary rainbow or an infinite series of fainter rainbows



The Mathematics of rainbows:

To begin the elementary mathematical treatment of rainbows, we present a brief derivation of a generic planar form of Snell's law of refraction, much used in connection with rainbows and halos, among other optical phenomena. What follows is usually stated in calculus texts as a land- and-water problem: an individual on land desires to reach a specific point out to sea in the shortest possible time, perhaps to rescue someone from the encroaching tide, or just because it's there.

However supposed, it is the mathematical equivalent of Fermat's principle of least time or more correctly, the statement that the path of the ray of light will be such that the total time of travel is constant from figure 4, the points A and B are fixed, and it is desired to find the point C such that the time T taken to travel the path ACB, composed of line segments AC and CB is a minimum.

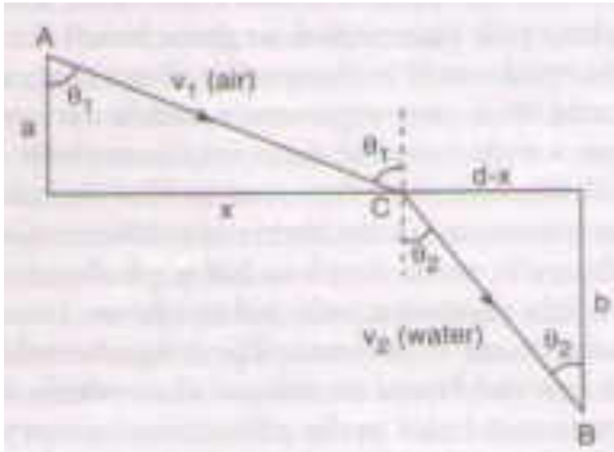


Fig 4

Let T_{AC} and T_{CB} be the times to travel those respective segments, so that with a, b and d constants,

$$T = T_{AC} + T_{CB} = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}, \qquad 0 < x < d.$$

Taking derivative, we get

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{(d-x)}{v_2\sqrt{b^2 + (d-x)^2}}, = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

which is of course zero at a critical point. This means that we may state the generic form of Snell's law, which states that

$\sin\theta_1 = n\sin\theta_2$ where $n = \frac{v_1}{v_2}$ will be referred as the refractive index.

In the context of meteorological optics, with light ray entering water droplets from air, the speeds v_1 and v_2 will represent the respective speeds of light in air and in water. Of course thus far we have not established that T is indeed a minimum time, but, by the first derivative test applied on either side of the critical value x_c for which $T''(x_c) = 0$,

$x \downarrow \Rightarrow \sin\theta_1 \downarrow \Rightarrow \sin\theta_2 \uparrow \Rightarrow T'' < 0$ and

$x \uparrow \Rightarrow \sin\theta_1 \uparrow \Rightarrow \sin\theta_2 \downarrow \Rightarrow T'' > 0$,

so this is indeed a minimum time. Now we will examine the primary rainbow. From figure note that after two refraction and one reflection the light ray shown contributing to the rainbow has undergone a total deviation of $D(i)$ radians, where

$$D(i) = \pi + 2i - 4r(i)$$

in terms of the angles of incidence i and reflection r, respectively.

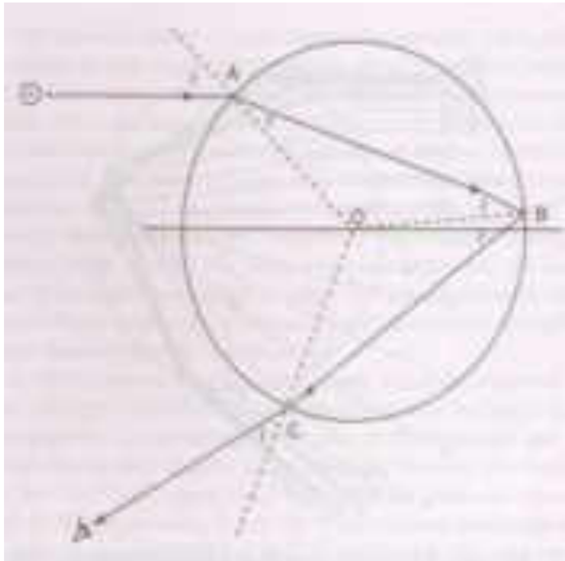


Fig 5

The latter is a function of the former, this relation being expressed in terms of Snell's law,
$\sin i = n \sin r,$
where n is the relative index of refraction. This relative index is defined as
$$n = \frac{speed\ of\ light\ in\ medium\ I\ (air)}{speed\ of\ light\ in\ medium\ II\ (water)} > 1.$$
Since the speed of light in air is almost that 'in vacuum' we will refer to n for simplicity as the refractive index; its generic value for water is $n \approx {}^4/_3$, but it does depend slightly on wavelength.

Now let us examine the behaviour of deviation of incident ray

i.e. $D(i) = \pi + 2i - 4\,r(i)$ Note that $D(0) = \pi$.

Differentiating, we get
$$\frac{dD}{di} = 2 - 4\frac{dr}{di},$$
again differentiating the Snell's law $\sin i = n \sin r$, we get
$$\cos i = n \cos r \frac{dr}{di},$$
so that
$$\frac{dD}{di} = 2 - \frac{4 \cos i}{n \cos r},$$
There is a critical number in the domain $i \in [0, \pi/2]$ for which $D'(i) = 0$.

Implies that
$$\frac{1}{4} = \frac{\cos^2 i}{n^2 \cos^2 r}$$
$$\Rightarrow \quad \frac{1}{4} = \frac{\cos^2 i}{n^2(1 - \sin^2 r)}$$

$$\Rightarrow \quad \frac{1}{4} = \frac{\cos^2 i}{n^2\left(1 - \frac{\sin^2 i}{n^2}\right)}$$

$$\Rightarrow \quad \frac{1}{4} = \frac{\cos^2 i}{n^2 - 1 + \cos^2 i}$$

again solving this we get that

$$\cos^2 i - \frac{\cos^2 i}{4} = \frac{n^2 - 1}{4}$$

$$\cos i = \sqrt{\frac{n^2 - 1}{3}} \equiv \cos i_c$$

For k internal reflections the corresponding result is

$$\cos i_c = \sqrt{\frac{n^2 - 1}{k(k+1)}}$$

as is readily verified.

we continue with case $k = 1$. For a generic monochromatic rainbow the choice $n = {}^4/_3$ yields

$$i_c \approx arccos\sqrt{{}^7/_{27}} \approx 59.4^0$$

and $D(i)$ may be written as

$$D(i_c) = \pi + 2i - 4\, arcsin\left(\frac{\sin i_c}{n}\right)$$

$$= 180^0 + 2(59.4^0) - 4\arcsin\left(\frac{\sin 59.4^0}{4/3}\right)$$

$$D(i_c) \approx 138^0$$

The supplement of this angle, $180^0 - D(i_c) \approx 42^0$, is the semi-angle of the rainbow cone formed with apex at the observer's eye, the axis being along the line joining the eye to the antisolar point.

For secondary rainbow angle of Deviation D is a function of two variables angle of incidence i and the refractive index n i. e. $D = D(i, n)$ then we can show that $D(i_c) \approx 129^0$

The supplementary angle of 129^0 is 51^0 at which we see secondary rainbow, i. e. it is about 9^0 higher in the sky than the primary rainbow.

In general for k internal reflection, it can be shown that the angle of deviation is

$$D(i) = k\pi + 2i - 2(k+1)\, arcsin\left(\frac{\sin i}{n}\right)$$

then interesting trigonometric calculations may be done for higher order rainbows.

Conclusion:

The most of the people know the beautiful phenomena rainbow. Our aim is to study rainbow in terms of mathematics. We studied different references and mathematically verified that the angle of elevation from the observer to the top of the rainbow being about 42 degree.

References:

1. John A. Adam, 2010, 'Mathematics in Nature', University press, India, p 88-91.

2. Rachel W. Hall, Nigel Higson, 1998, 'The Calculus of Rainbows'.
3. Thao Dang, 'The Theory of The Rainbows', Physics 464, Project report.